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# A Picone Identity for First Order Differential Systems*

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The Picone identity for solutions of Sturm–Liouville equations is generalized to solutions of certain first order nonlinear differential inequalities, to first order vector and matrix systems, and to certain first order systems of partial differential equations. These identities lead to generalizations of the Sturm–Picone Theorem, a disconjugacy criterion for a nonselfadjoint fourth order differential equation, and to a generalized maximum principle for elliptic equations.

## 1. INTRODUCTION

Picone's identity deals with functions $u(x)$ and $v(x)$ which are nontrivial solutions of Sturm-Liouville equations

$$(au')' + cu = 0, \tag{1.1}$$

$$(gv')' + hv = 0, \tag{1.2}$$

respectively. If $u(x)$ and $v(x)$ are of class $C^2$ and $v(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{u}{v}(au'v - gv'u)\right] = u(au')' - \frac{u^2}{v}(gv')' + (a-g)\,u'^2 + g\left(u' - u\frac{v'}{v}\right)^2.$$

Making use of the fact that $u(x)$ and $v(x)$ also satisfy (1.1) and (1.2), respectively, one obtains

$$\frac{d}{dx}\left[\frac{u}{v}(au'v - gv'u)\right] = (h-c)\,u^2 + (a-g)\,u'^2 + g\left(u' - u\frac{v'}{v}\right)^2. \tag{1.3}$$

From (1.3) one easily obtains the Sturm-Picone Theorem [1]: if $a \geqslant g > 0$ and $h \geqslant c$, then solutions of (1.2) oscillate faster than solutions of (1.1) in the sense that zeros of $v(x)$ separate the zeros of $u(x)$.

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The Sturm-Picone Theorem is amenable to numerous generalizations, and several such generalizations have been derived from a more general form of (1.3). In particular, identities similar to (1.3) have been used to establish comparison theorems for elliptic equations by Picone [2] and the author [3] and for second order ordinary and elliptic systems by the author [4], [5].

The purpose of this paper is to study first order systems of differential equations by means of an identity similar to (1.3). For example, in the simplest scalar case we shall consider the systems

$$u' = ew; \qquad w' = -cu \tag{1.4}$$

and

$$v' = fz; \qquad z' = -hv \tag{1.5}$$

where $e(x) \geqslant 0$ and $f(x) \geqslant 0$. In case $e(x) > 0$ and $f(x) > 0$, the systems (1.4) and (1.5) are equivalent to (1.1) and (1.2), respectively. However, by only assuming that $e(x)$ and $f(x)$ are nonnegative, the first order systems to be considered will be more general than the second order cases previously handled. This generalization is especially important in the case of the vector valued systems of the form (1.4) and (1.5) which are studied in Section 3 and can be used to represent certain differential equations of order $2n$. In such a representation the matrix analogues of $e(x)$ and $f(x)$ are only positive semi-definite. As an application of the Sturm-Picone Theorem for vector systems established in Section 3 we shall give a new disconjugacy criterion for a non-selfadjoint fourth order differential equation. As an application of the generalized Picone identity for first order partial differential systems established in Section 4, a maximum principle for such systems is established.

The techniques used below also have the advantage of applying to non-linear differential inequalities with the same ease to which they apply to (1.4) and (1.5). This fact is brought out in Section 2, but for the sake of simplicity is not fully exploited in the remaining sections.

## 2. The Scalar Case

In this section $u$, $v$, $w$, and $z$ will denote real valued functions of $x$ which are continuously differentiable. The functions $c(x, u, w)$, $e(x, u, w)$, $f(x, v, z)$, and $h(x, v, z)$ are assumed to be continuous in $x$ for all values of the other variables, and in addition $e$ and $f$ are assumed to be nonnegative. We shall consider the systems

$$u' = ew; \qquad uw' \geqslant -cu^2, \tag{2.1}$$

$$zv' \geqslant fz^2; \qquad vz' \leqslant -hv^2 \tag{2.2}$$

which are generalizations of the Sturm-Liouville equations $(au')' + cu = 0$ and $(gv') + hv = 0$. Our first result is a generalization of the Sturm-Picone Theorem to (2.1) and (2.2).

THEOREM 2.1. *Suppose there exists a function $w(x)$ such that $u(x)$ is a nontrivial solution of* (2.1) *and that $u(x_1) = u(x_2) = 0$ for some $x_1 < x_2$. If there exists a function $z(x)$ such that $v(x)$ satisfies* (2.2) *and*

(i) $f(x, v, z) \geqslant e(x, u, w) \geqslant 0$,

(ii) $\int_{x_1}^{x_2}(h - c)\, u^2\, dx > 0$,

*then $v(x)$ has a zero in $[x_1, x_2]$.*

*Proof.* Suppose $v(x) \neq 0$ in $[x_1, x_2]$. Then a direct calculation verifies that

$$\frac{d}{dx}\left[\frac{u}{v}(wv - zu)\right] = uw' - \frac{u^2}{v} z' + wu' - 2\frac{zuu'}{v} + \frac{zu^2v'}{v^2}.$$

Making use of (2.1) and (2.2) and inserting

$$0 = -\frac{e^2}{f} w^2 + \frac{e^2}{f} w^2$$

yields

$$\frac{d}{dx}\left[\frac{u}{v}(wv - zu)\right] \geqslant (h - c)\, u^2 + \left(e - \frac{e^2}{f}\right) w^2 + \left(\frac{e}{\sqrt{f}} w - \sqrt{f}\,\frac{uz}{v}\right)^2, \tag{2.3}$$

where (i) assures that $(e^2/f)$ is well-defined.

Integrating (2.3) from $x_1$ to $x_2$ and making use of $u(x_1) = u(x_2) = 0$ we get

$$0 \geqslant \int_{x_1}^{x_2} \left[(h - c)\, u^2 + e\left(1 - \frac{e}{f}\right) w^2\right] dx \tag{2.4}$$

which contradicts the hypotheses and shows that $v(x) = 0$ for some $x$ in $[x_1, x_2]$.

*Remarks.*

1. In case $v' = fz$ in (2.2), we have equality in (2.4) only if

$$\int_{x_1}^{x_2} \left(\frac{e}{\sqrt{f}} w - \sqrt{f}\,\frac{uz}{v}\right)^2 dx = \int_{x_1}^{x_2} \frac{1}{\sqrt{f}}\left(u' - \frac{uv'}{v}\right)^2 dx = 0$$

—i.e., only if $u' \equiv (uv'/v)$ which implies that $v$ is a constant multiple of $u$. Thus in this case we can relax (ii) to

$$\int_{x_1}^{x_2} (h - c)\, u^2\, dx \geqslant 0.$$

**2.** In the special case where (2.1) and (2.2) are replaced by linear equalities and

$$f(x) = \frac{1}{g(x)} \geqslant e(x) = \frac{1}{a(x)} > 0,$$

Theorem 2.1 reduces to the classical Sturm-Picone Theorem for (1.1) and (1.2).

**3.** It is also of interest to consider the nonlinear systems

$$u' = ew; \qquad w' = -cu \tag{2.5}$$

$$zv' \geqslant fz^2; \qquad vz' \leqslant -hv^2 \tag{2.6}$$

under the additional hypotheses that $c(x)$ and $h(x)$ be nonpositive. Then the assumption $z(x) \neq 0$ leads to the complementary inequality

$$\frac{d}{dx}\left[\frac{w}{z}(wv - zu)\right] \geqslant (f - e)\, w^2 + \left(c - \frac{c^2}{h}\right) u^2. \tag{2.7}$$

If in addition $w(x_1) = w(x_2) = 0$ and

(i) $h(x, v, z) \geqslant c(x, u, w)$,

(ii) $\int_{x_1}^{x_2} (f - e)\, u^2\, dx > 0$,

then $z(x)$ has a zero in $[x_1, x_2]$. In the special case of the Sturm-Liouville equations (1.1) and (1.2) with $h(x) \geqslant c(x) > 0$ and $g(x) \geqslant a(x) > 0$, this reduces to Leighton's observation [6, Theorem 3] that the zeros of $v'(x)$ separate the zeros of $u'(x)$.

The identity (2.3) also leads to generalization of comparison theorems of the type considered by Grimmer and Waltman [7] for nonlinear differential inequalities of the form

$$v'' + k(x)\, v' + l(x, v)\, v \geqslant 0,$$

$$y'' + k(x)\, y' + l(x, y)\, y \leqslant 0.$$

Multiplying through by $e^{k(x)}$ these inequalities can be put into selfadjoint form

$$(av')' + c(x, v)\, v \geqslant 0,$$

$$(ay')' + c(x, y)\, y \leqslant 0,$$

which in turn constitute a special case of the systems

$$y' = e(x)\,t; \qquad t' \geqslant -\,c(x, y)\,y, \tag{2.8}$$

$$v' = e(x)\,z; \qquad z' \leqslant -\,c(x, v)\,v, \tag{2.9}$$

where $e(x)$ is only assumed to be nonnegative.

THEOREM 2.2. *Suppose there exist functions $z(x)$ and $t(x)$ such that $y(x)$ and $v(x)$ satisfy (2.8) and (2.9), respectively, with $v(x_1) = y(x_1) > 0$, $y'(x_1) > v'(x_1) > 0$. If $c(x, y)$ is nonincreasing in $y$ for nonnegative $y$ then as long as $v(x)$ and $y(x)$ exist and $v(x) > 0$, $y(x) > v(x)$.*

*Proof.* Suppose, to the contrary, that there exists $x_2 > x_1$ such that $y(x_2) = v(x_2) > 0$ while $v(x) > 0$ for $x_1 \leqslant x \leqslant x_2$ and $y(x) > v(x)$ for $x_1 < x < x_2$. Defining $u = y - v$ we have $u(x_1) = u(x_2) = 0$, $u'(x_1) > 0$, and $u(x) > 0$ for $x_1 < x < x_2$. Subtracting (2.9) from (2.8) yields

$$u' = e(x)\,w; \qquad w' \geqslant c(x, v)\,v - c(x, y)\,y,$$

where $w(x) = t(x) - z(x)$. Furthermore since $c(x, y)$ is nonincreasing in $y$, the above inequality implies that

$$w' \geqslant -\,c(x, y)\,(y - v) - v[c(x, y) - c(x, v)]$$
$$w' \geqslant -\,c(x, y)\,u \geqslant -\,c(x, u)\,u$$

so that $u(x)$ satisfies

$$u' = e(x)\,w; \qquad w' \geqslant -\,c(x, u)\,u. \tag{2.10}$$

Applying Theorem 2.1 and the subsequent Remark 1 to (2.10) and (2.9), we obtain the contradiction that $v(x)$ has a zero in $[x_1, x_2]$.

## 3. VECTOR AND MATRIX SYSTEMS

In this section $\mathbf{u}$ and $\mathbf{w}$ will denote real $n \times 1$ column vectors whose components are continuously differentiable functions of $x$. Capital letters $U$, $V$, $W$, and $Z$ will denote real $n \times n$ matrices whose components are also continuously differentiable. The real $n \times n$ matrices $C(x, \cdot, \cdot)$, $E(x, \cdot, \cdot)$, $F(x, \cdot, \cdot)$, and $H(x, \cdot, \cdot)$ are assumed to have real components which are continuous in $x$, and $E$ and $F$ are assumed to be positive semidefinite. In analogy to Section 2, we shall consider the systems

$$\mathbf{u}' = E\mathbf{w}; \qquad \mathbf{w}' = -\,C\mathbf{u} \tag{3.1}$$

$$V' = FZ; \qquad Z' = -\,HV. \tag{3.2}$$

While it is possible to consider differential inequalities in place of (3.1) and (3.2) analogous to those in (2.1) and (2.2), we shall not pursue this generalization here.

In case $F$ and $H$ are symmetric and $V$, $Z$ is a solution of (3.2), it is easy to verify by differentiation that $V^*Z - Z^*V$ is constant. Of special interest will be those solutions of (3.2) for which $V^*Z - Z^*V$ is zero.

DEFINITION 3.1. A matrix solution $V$, $Z$ of (3.2) is called a *conjugate solution* if $V^*Z = Z^*V$.

For a discussion of conjugate solutions of matrix differential equations see Sternberg [8] or Barrett [9].

DEFINITION 3.2. Given two positive semidefinite matrices $E$ and $F$ we write $E \prec F$ if

(i) The range of $E$ is orthogonal to the null space of $F$, and

(ii) $E^* - E^*F^{-1}E \geqslant 0$,

where $F^{-1}$ is to be interpreted as the inverse of $F$ in the range of $E$ and inequalities are to be interpreted in the sense of positive definiteness.

THEOREM 3.3. *Suppose the vectors* $\mathbf{u}(x)$, $\mathbf{w}(x)$ *are a nontrivial solution of* (3.1) *satisfying* $\mathbf{u}(x_1) = \mathbf{u}(x_2) = 0$ *for some* $x_1 < x_2$. *Suppose also that F and H are symmetric. If* $V$, $Z$ *is a conjugate system for* (3.2) *and if*

(i) $E(x, \mathbf{u}, \mathbf{w}) \prec F(x, V, Z)$ *for* $x_1 \leqslant x \leqslant x_2$

(ii) $\int_{x_1}^{x_2} \mathbf{u}^*(H - C)\,\mathbf{u}\,dx \geqslant 0$,

*then* $\det V \equiv |V|$ *has a zero in* $[x_1, x_2]$.

*Proof.* Suppose $|V(x)| \neq 0$ in $[x_1, x_2]$. Then a direct calculation shows that

$$\frac{d}{dx}(\mathbf{u}^*\mathbf{w} - \mathbf{u}^*ZV^{-1}\mathbf{u}) = \mathbf{u}^*\mathbf{w}' - \mathbf{u}^*Z'V^{-1}\mathbf{u} + \mathbf{u}^{*\prime}\mathbf{w} - \mathbf{u}^{*\prime}ZV^{-1}\mathbf{u}$$
$$- \mathbf{u}^*ZV^{-1}\mathbf{u}' + \mathbf{u}^*ZV^{-1}V'V^{-1}\mathbf{u}.$$

Here we have used the relation $(V^{-1})' = V^{-1}V'V^{-1}$. Making use of (3.1) and (3.2) and inserting $0 = -\mathbf{w}^*E^*F^{-1}E\mathbf{w} + \mathbf{w}^*E^*F^{-1}E\mathbf{w}$ yields

$$\frac{d}{dx}(\mathbf{u}^*\mathbf{w} - \mathbf{u}^*ZV^{-1}\mathbf{u}) = \mathbf{u}^*(H - C)\,\mathbf{u}$$
$$+ \mathbf{w}^*E^*\mathbf{w} - \mathbf{w}^*E^*F^{-1}E\mathbf{w} + \mathbf{w}^*E^*F^{-1}E\mathbf{w}$$
$$- \mathbf{w}^*E^*ZV^{-1}\mathbf{u} - \mathbf{u}^*ZV^{-1}E\mathbf{w} + \mathbf{u}ZV^{-1}FZV^{-1}\mathbf{u}.$$

Making use of the symmetry of $F$ and the fact that the conjugacy property $V^*Z = Z^*V$ implies that $ZV^{-1} = V^{-1*}Z^*$, we obtain

$$\begin{aligned}\frac{d}{dx}(\mathbf{u}^*\mathbf{w} - \mathbf{u}^*ZV^{-1}\mathbf{u}) &= \mathbf{u}^*(H - C)\,\mathbf{u} + \mathbf{w}^*(E^* - E^*F^{-1}E)\,\mathbf{w} \\ &\quad + (\sqrt{F^{-1}}E\mathbf{w} - \sqrt{F}ZV^{-1}\mathbf{u})^* \\ &\quad \times (\sqrt{F^{-1}}E\mathbf{w} - \sqrt{F}ZV^{-1}\mathbf{u}).\end{aligned} \tag{3.3}$$

Integrating (3.3) from $x_1$ to $x_2$ yields

$$0 \geqslant \int_{x_1}^{x_2} \mathbf{u}^*(H - C)\,\mathbf{u}\,dx + \int_{x_1}^{x_2} \mathbf{w}^*(E^* - E^*F^{-1}E)\,\mathbf{w}\,dx \tag{3.4}$$

with equality iff

$$\sqrt{F^{-1}}E\mathbf{w} - \sqrt{F}ZV^{-1}\mathbf{u} \equiv 0$$

or

$$\mathbf{u}' - V'V^{-1}\mathbf{u} \equiv 0.$$

Operating on this equation with $V^{-1}$ yields

$$\frac{d}{dx}(V^{-1}\mathbf{u}) \equiv 0$$

which implies that we have equality in (3.4) iff $\mathbf{u} = V\mathbf{k}$ for some constant vector $\mathbf{k}$, so that equality in (3.4) implies that $| V | = 0$ at $x = x_1$ and $x_2$. On the other hand inequality in (3.4) contradicts the hypotheses (i) and (ii), and this shows that $| V | = 0$ for some $x$ in $[x_1, x_2]$.

*Remarks.*

1. The system (3.1) can be replaced by a matrix system by substituting matrices $U$ and $W$ for the vectors $\mathbf{u}$ and $\mathbf{w}$.

2. In the proof of Theorem 3.3, we only require the symmetry of $F(x, V, Z)$. However, in order to make use of the theory of conjugate solutions in applying this result one must also require the symmetry of $H(x, V, Z)$.

3. When specialized to linear second order vector equations, Theorem 3.3 reduces to a criterion for disconjugacy due to Hartman and Wintner [10, Theorem II].

As mentioned earlier, first order vector and matrix systems like (3.1) and (3.2) enter into the study of higher order ordinary differential equations. As an application of Theorem 3.3 to such problems we shall consider the nonsingular fourth order equations

$$[(au'')' + pu']' + bu' + cu = 0 \tag{3.5}$$

$$[(gv'')' + qv']' + hv = 0 \tag{3.6}$$

and establish disconjugacy criteria for (3.5).

DEFINITION 3.4. We say that (3.5) is disconjugate on $[x_1, x_2]$ if there exists no $x_0$ in $(x_1, x_2]$ such that (3.5) has a nontrivial solution satisfying $u(x_1) = u'(x_1) = u(x_0) = u'(x_0) = 0$.

For the sake of simplicity (and without loss of generality) we shall take $x_1 = 0$. In order to transform (3.5) and (3.6) into appropriate first order systems we make use of a transformation due to Sternberg [8] (see also Barrett [11]). Define

$$\mathbf{u} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix}; \qquad \mathbf{w} = \begin{pmatrix} 0 & -1 \\ 1 & -x \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

where $u_1 = au''$ and $u_2 = u_1' + pu'$. Then (3.5) becomes

$$\mathbf{u}' = E\mathbf{w}; \qquad w' = -C\mathbf{u} \tag{3.7}$$

where

$$E = \frac{1}{a}\begin{pmatrix} x^2 & -x \\ -x & 1 \end{pmatrix}; \qquad C = -c\begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} + \begin{pmatrix} 0 & -b \\ 0 & p - bx \end{pmatrix}.$$

Similarly defining

$$\mathbf{v} = \begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ v' \end{pmatrix}; \qquad \mathbf{z} = \begin{pmatrix} 0 & -1 \\ 1 & -x \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

where $v_1 = gu''$ and $v_2 = v_1' + qv'$, (3.6) becomes

$$\mathbf{v}' = F\mathbf{z}; \qquad \mathbf{z}' = -H\mathbf{v} \tag{3.8}$$

where

$$F = \frac{1}{g}\begin{pmatrix} x^2 & -x \\ -x & 1 \end{pmatrix}; \qquad H = -h\begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}.$$

We note that $F$ and $H$ are symmetric and that $E$ and $F$ are positive semidefinite. Furthermore

$$H - C = (c - h)\begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} + \begin{pmatrix} 0 & -b \\ 0 & q - p + bx \end{pmatrix}$$

is positive semidefinite whenever $c - h \geqslant 0$ and $(c - h)(q - p) - b^2/4 \geqslant 0$, and $E \prec F$ whenever $a \geqslant g > 0$.

Consider now the matrix system

$$V' = FZ; \qquad Z' = -HV \tag{3.9}$$

associated with 3.8. Applying Theorem 3.3 to the systems (3.7) and (3.9), we have the following result.

THEOREM 3.4. *Suppose* (3.5) *has a nontrivial solution satisfying* $u(0) = u'(0) = u(x_0) = u'(x_0) = 0$. *If* $c - h \geqslant 0$, $(c - h)(q - p) - b^2/4 \geqslant 0$ *on* $(0, x_0)$, *and if* $V$, $Z$ *is a conjugate system for* (3.9), *then* $|V|$ *has a zero in* $[0, x_0]$.

Specific disconjugacy criteria for the nonselfadjoint equation (3.5) can now be obtained by noting [11] that (3.6) is disconjugate on $[0, x_0]$ iff the solution of

$$\begin{aligned} V' &= FZ; \qquad Z' = -HV \\ V(0) &= 0; \qquad Z(0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{aligned} \tag{3.10}$$

satisfies $|V| \neq 0$ on $(0, x_0]$. This fact leads to the following comparison theorem, generalizing a similar theorem of Barrett [11, Theorem 4.1] (which deals only with the case $b \equiv 0$).

COROLLARY 3.5. *If* (3.6) *is disconjugate on* $[0, x_0]$ *and if* $c - h \geqslant 0$, $(c - h)(q - p) - b^2/4 \geqslant 0$ *on* $(0, x_0)$, *then* (3.5) *is also disconjugate on* $[0, x_0]$.

Since conditions under which the selfadjoint equation (3.6) is disconjugate are well-known, Corollary 3.5 can be used to establish specific criteria for the disconjugacy of (3.5).

## 4. ELLIPTIC EQUATIONS

The identity (2.3) also has a natural generalization to elliptic equations or, more generally, to the first order systems

$$\frac{\partial u}{\partial x_i} = \sum_{j=1}^{n} e_{ij} w_j; \qquad \sum_{j=1}^{n} \frac{\partial w_j}{\partial x_j} = -cu, \tag{4.1}$$

$$\frac{\partial v}{\partial x_i} = \sum_{j=1}^{n} f_{ij} z_j; \qquad \sum_{j=1}^{n} \frac{\partial z_j}{\partial x_j} = -hv, \tag{4.2}$$

which can also be written

$$\nabla u = E\mathbf{w}; \qquad \nabla \cdot \mathbf{w} = -cu, \tag{4.1$'$}$$

$$\nabla v = F\mathbf{z}; \qquad \nabla \cdot \mathbf{z} = -hv, \tag{4.2$'$}$$

respectively. In the notation of the latter formulation $u$ and $v$ will denote real valued functions of $x = (x_1, ..., x_n)$ which are continuously differentiable while $\mathbf{w}$ and $\mathbf{z}$ will denote vectors whose components are continuously differentiable functions of $x$. The functions $c(x, u, \mathbf{w})$ and $h(x, v, \mathbf{z})$ are assumed continuous in $x$ for all values of the other variables, and $E(x, u, \mathbf{w})$ and $F(x, v, \mathbf{z})$ are to be positive semidefinite symmetric $n \times n$ matrices whose elements are continuous in $x$ for all values of the other variables. Thus the systems (4.1) and (4.2) are generalizations of the selfadjoint elliptic equations

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial u}{\partial x_i}\right) + cu = 0,$$

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(g_{ij}\frac{\partial v}{\partial x_i}\right) + hv = 0.$$

Our first observation is that the Sturmian comparison theorem for such elliptic equations [3] has a direct generalization to the systems (4.1) and (4.2). While we could again deal with differential inequalities as in Section 2, we shall formulate our results in terms of the systems of equations (4.1) and (4.2).

THEOREM 4.1. *Suppose there exist functions* $u(x)$, $\mathbf{w}(x)$ *satisfying* (4.1) *in a sufficiently smooth bounded domain* $D \subset R^n$ *and that* $u(x) \neq 0$ *in* $D$, $u(x) = 0$ *on* $\partial D$. *If* $v(x)$, $\mathbf{z}(x)$ *satisfy* (4.2) *and*

(i) $E(x, u, \mathbf{w}) \prec F(x, v, \mathbf{z})$ *for all* $x$ *in* $D$,

(ii) $\int_D (h - c)\, u^2\, dx \geqslant 0$,

*then* $v(x)$ *has a zero in* $\bar{D}$.

*Proof.* If $v(x) \neq 0$ in $\bar{D}$, then we have the following direct analogue to (2.3):

$$\nabla \cdot \left[\frac{u}{v}(v\mathbf{w} - u\mathbf{z})\right] = (h - c)\, u^2 + \mathbf{w}^*(E^* - E^*F^{-1}E)\,\mathbf{w} \\ + \left(\sqrt{F^{-1}}E\mathbf{w} - \frac{u}{v}\sqrt{F}\mathbf{z}\right)^* \left(\sqrt{F^{-1}}E\mathbf{w} - \frac{u}{v}\sqrt{F}\mathbf{z}\right). \tag{4.3}$$

Integrating (4.3) over $D$ and applying Green's Theorem yields

$$0 \geqslant \int_D (h - c)\, u^2\, dx + \int_D \mathbf{w}^*(E^* - E^*F^{-1}E)\,\mathbf{w}\, dx \tag{4.4}$$

with equality iff

$$\sqrt{F^{-1}}E\mathbf{w} = \frac{u}{v}\sqrt{F}\mathbf{z}$$

or

$$\nabla u = \frac{u}{v}\nabla v. \tag{4.5}$$

However (4.5) is satisfied iff $v$ is a constant multiple of $u$, in which case $v = 0$ on $\partial D$. If (4.5) is not satisfied then we have inequality in (4.4) which contradicts the hypotheses (i) and (ii).

As a final application of such first order Picone identities we shall establish the following maximum principle for solutions of first order systems of the form (4.1) for which $E = E(x)$.

THEOREM 4.2. *Suppose $E$ is independent of $u$ and* $\mathbf{w}$ *and that $u(x)$,* $\mathbf{w}(x)$ *are solution of* (4.1) *in a domain $D$ in which $c(x, u, \mathbf{w}) \leqslant 0$. If $u(x)$ has a positive maximum at $x_0 \in D$, then $u(x)$ is constant in a neighborhood of $x_0$.*

*Proof.* Consider the system (4.2) with $F \equiv E$ and $h(x, v, \mathbf{z}) \equiv 0$. This system has the obvious solution $v \equiv 1$, $\mathbf{z} \equiv \mathbf{0}$. Now if $u(x)$ has a positive maximum at $x_0 \in D$, then there exists a proper subdomain $D' \subset D$ such that $u(x)$ is positive in $\bar{D}'$ and $\boldsymbol{\nu}^*\mathbf{w} \leqslant 0$ on $\partial D'$, where $\boldsymbol{\nu}$ denotes the exterior normal to $D'$. Integrating (4.3) over $D'$ and applying Green's Theorem yields

$$\int_{\partial D'} \boldsymbol{\nu}^*(u\mathbf{w})\, d\sigma \geqslant -\int_{D'} cu^2\, dx$$

with equality iff $v$ is a constant multiple of $u$—i.e., iff $u(x)$ is a constant. Since our hypotheses preclude inequality, it follows that $u \equiv$ constant in $D'$.

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